On One Model of the Geometrical Quintessence

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Abstract

A cosmological model with modification of the Einstein-Hilbert action by the correction $f(R) \propto \beta R^n$ is considered. Such way of the description of the "geometrical" dark energy has been introduced repeatedly and the coefficients of the model were chosen to be fit against some observational data. In this paper the unambiguous choice of parameters n and β is proposed from the follow reasons: the exponent n close to 1.2953 follows from the request for the evolution of the Universe after recombination to be close to the evolution of the flat FRW model with cold dark matter and the reasonable age of the Universe defines the magnitude of the coefficient β. Such a model corresponding to the evolution of the Universe with the dynamical Λ-term describes well enough the observational data.

1   Introduction

The discovery of accelerated expansion of the Universe [1, 2, 3] has stimulated the quest for mechanism of the modern inflation. The most famous theoretical model of dark energy (DE) is the cosmological constant Λ. The corresponding FRW solution for flat Universe with the present densities ratio for cold matter and dark energy ($\Omega_m/\Omega_{\Lambda 0} \sim 0.3/0.7$) describes satisfactorily the evolution of the Universe at low redshifts [3, 4, 5]. However, the nature of the constant Λ- term has been remaining to be inexplicable during many years.

It is well-known, the application of the constant Λ- term for modeling of the early Universe was initially confronted with principal difficulties. Solving the problems of the very early Universe, this term has to be reducing by several order of magnitude during following evolution of the Universe. This problem was solved by rejection of Λ- term, and corresponding inflationary behavior was determined by models of " effective Λ- term"-quasi-classical scalar fields in the one way. The progress of these models is well-known. Another way to describe the inflationary behavior is to take into account the polarization of vacuum of quantum fields in the early Universe. Taking into account of the effects of the polarization leads to the appearance of the terms non-linear on curvature in the Einstein-Hilbert action. In such models the inflation appears self-consistently. Let us

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note two issues

- a correction to the Einstein -Hilbert action with the arbitrary function of scalar curvature R is equivalent mathematically to the introduction of scalar field into the classical Friedman cosmology[6, 7, 8];

- the terms of the form R^n were investigated in the early works done on the problem of singularity before obtaining of exact corrections to Einstein action following from the one loop approximation [9]. The part of such solutions approaches asymptotically to Friedman solutions with Λ-term, however physical results of the solutions have not been explained at that time.

For the purpose to explain the accelerated expansion of the Universe today, it is naturally to use the experience cumulative in investigations of early Universe. Thus, one of the tendencies is concerned with the hypothesis of existence some scalar fields that determine the density of dark energy (DE) (e.g. [10, 11, 12]). The another tendency models an effective quasi-hydrodynamical tensor of momentum-energy describing the observational data [13]. And third tendency is to generalize the Einstein-Hilbert equation by inclusion of curvature invariants [14, 15, 16] analogously to the more early works mentioned above. The last approach one can consider to be either an independent approach to the describing of DE or an analogue of inclusion of scalar fields (in the case $f(R)$) as stated above.

In the works on the higher order gravity theories (HOGT) the models with power corrections were investigated, however they have never been fitted to whole set of the observational data.

In this paper, the model with correction $f(R) \propto R^n$ with $n > 0$ is considered in detail for the purpose to correlate it with the observational data. In the other words, we would like to obtain the model that does not conflict with the scenario of the large scale structure formation (in past) and describes satisfactorily the Universe undergoing an accelerated expansion at present. Therefore, at the minimum, in the framework of the $f(R)$-theories, we will obtain solutions remind ΛCMD model describing well enough by set of the observational data [3, 4]. However, as it has been mentioned by various authors (see [5] and references therein) the observational data indicates the models of the dynamical DE. Hence, our second aim is to search out such dynamical solutions in the framework of the HOGT and to find out whether these solutions are preferable.

This paper is built as follows. In the section II the basic equations of the HOGT are presented. In the section III for the corrections of the form $f(R) \propto \beta R^n$ we find the exponent $n_1 = 1.2953$ which allows the generalized Einstein equation for the scale factor a to have a particular solution corresponding to the flat FRW solution for cold matter. We show that instability of the solutions that are close to this particular solution at $z >> 1$ may lead to the accelerated behavior of the model at present and the following asymptotic approach of the solution to the solution with the constant Λ- term. In the section IV we discuss our results and compare them to observations. In the model there are only two free parameters of the model - the coefficient β and a slight deviation of the parameter n from $n_1 = 1.2953$ mentioned above. Fixed from one set of the observational data they allows to obtain the rest of the set of the observational data.

2 Basic equations

We will work in the matter frame with a spatially flat FRW metric

$$ds^2 = d\tau^2 - a^2(\tau)(dx^2 + dy^2 + dz^2), \tag{1}$$

with $a(\tau)$ being a scale factor. The non-dimensional components of Ricci scalar and Ricci tensor are,

$$R = -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right), \quad R_0^0 = -3\left(\frac{\ddot{a}}{a}\right), \quad \dot{a} = \frac{1}{H_0}\frac{da}{d\tau}. \tag{2}$$

Here the present Hubble parameter H_0 has been introduced to turn to non-dimensional values. The equations that can be derived by variation of the action

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\,(R + f(R)) + \int d^4x \sqrt{-g}\, L_m \tag{3}$$

are well-known for a long time [17]. It is useful to reduce its $0-0$ component , that is the third order equation in a, to the second order equation by introduction of a new function

$$y(a) = (\dot{a}a)^2. \tag{4}$$

Thus, Eqs.(2) are determined as

$$R = -3y'/a^2, \quad R_0^0 = 3(y - a\,y')/2, \quad y' = dy/da, \tag{5}$$

and the observable parameters - the Hubble parameter h and the deceleration parameter q - are determined by formulae

$$h(a) = \sqrt{\frac{y}{a^4}}, \quad q = -\frac{\ddot{a}a}{(\dot{a})^2} = \left(1 - \frac{d\ln(y)}{d\ln(a^2)}\right). \tag{6}$$

Further, the $0-0$ component of Einstein equation can be represented in the form

$$y + \left[\frac{df}{dR}\left(y - \frac{a}{2}\frac{dy}{da}\right) - \frac{a^4}{6}f + a\,y\frac{d}{da}\left(\frac{df}{dR}\right)\right] = \Omega_m\,a. \tag{7}$$

The solution for the classical Friedman model with cold matter is defined by $f(R) = 0$ whereas for the model with the constant Λ-term it is defined by $f(R) = const$. Taken together they describe ΛCDM model, so, according to Eqs.(5),(7) we have

$$y = \Omega_m\,a + \Omega_\Lambda\,a^4, \quad \Omega_m = \rho_m/\rho_c, \quad \Omega_\Lambda = \rho_\Lambda/\rho_c, \tag{8}$$

where ρ_m, ρ_Λ, and $\rho_c = 3H_0^2/8\pi G$ are the cold matter density, the dark energy density, and the critical density of FRW Universe at present, respectively. For 4-interval (1) $\Omega_m + \Omega_\Lambda = 1$. The present-day observational values are close to $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$.

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Notice that the important property of Eq.(7) is a possible asymptotic exit of solutions to regime of an exponential expansion. Indeed, according to definition (8) this solution is $y = \Omega\, a^4$. In this case, the scalar curvature approaches to constant $R_\infty \to -12\Omega$. In the case of a negligible effect of matter this solution is a particular solution of Eq.(7) at execution of the following condition

$$\Omega = \frac{f(R_\infty)}{6(1 - (df/dR)|_\infty)} = const > 0. \tag{9}$$

3 Model with Power Corrections in Einstein -Hilbert Action

In this section we will investigate corrections to the Einstein-Hilbert action of the form

$$f(R) = -\alpha\, R^n. \tag{10}$$

In such a case, Eq. (4) can be presented in the form

$$\beta\left[n(n-1)y''y + \frac{(1-n)}{2}(y')^2 + n(4-3n)\frac{y'y}{a}\right] = \frac{(y')^{2-n}}{a^{4-3n}}(y - \Omega_m\, a),$$
$$\beta = (-3)^{n-1}\alpha. \tag{11}$$

A general approach to investigation of the last equation is given in paper[18].

There are two parameters in the model, β and n, determined by observational data. In a number of papers these parameters were chosen according to different requirements [15, 16]. In the present paper we will choose parameter n from requirement of closeness of evolution of the our model to the classical solution for the flat FRW Universe with cold matter in the past. This fact allows this scenario to be close to the scenario of the large scale structure formation. This requirement can be realized on condition the classical Friedman solution

$$y = \Omega_m\, a \tag{12}$$

presents a particular solution of Eq.(11). It is easy to see from Eq.(11), the last condition is equivalent to the choice of n to be satisfying the equation

$$n - 1 = 2n(4 - 3n) \quad \text{with roots} \quad n_1 = 1.295, \quad n_2 = -0.129. \tag{13}$$

The first of the roots leads to the type of models of papers [15, 16], while the second root corresponds the models with correction of the form $\propto \mu/R^{|n_2|}$ investigated in the paper [20]. As it will be shown below, such a choice of n approaches the model to the set of the observational data in the best way.

3.1 Behavior of the dust solution in the $f(R)$-model in past

After recombination, the evolution of the Universe has to be described by the Friedman model with the cold matter. The dynamics of expansion is determined by stability of the dust solution in the model (11). If the solution is stable, then the model evolves in the way very close to classical one. However, the observational data at $z < 1$ does not correspond to such a scenario, i.e. we are interested in dust-like solutions which are not stable in the model (11) but the perturbations do not grow catastrophically fast. Otherwise, the model does not provides a sufficiently long period with $q \simeq 0.5$ at $z \gg 1$ required for the large structure formation.

For investigation of behavior of the solution (12) we will search a perturbed solution in the form

$$y = \Omega\, a(1 + \psi), \quad \psi \ll 1. \tag{14}$$

In this linear approximation and close to the recombination time ($a \ll 1$) Eq. (11) yields,

$$n\, a^2\, \psi'' + a\, \psi'(2n - 0.5) = 0, \tag{15}$$

with the damping solution $\psi = C_1 + C_2\, a^{(\frac{1}{2n} - 1)}$. Hence the solution (12) for $f(R)$-theory (10) with $n = n_1$ satisfying Eq.(13) asymptotically approaches to to the flat FRW solution, i.e. it is stable and does not satisfies the requirement stated above[1]

As the next step on the path of the choice of n, we will look for solutions of the $f(R)$ -theories (10) with n which is a little different from n_1,

$$n = n_1 + \delta n, \qquad \delta n \ll 1. \tag{16}$$

It is efficient to rewrite the condition (13) in the form

$$\delta\, (n - 1) = 2n(4 - 3n), \quad \delta = 1 + \epsilon, \quad \epsilon \ll 1. \tag{17}$$

We will look for the perturbed dust-like solution in the form (14) near to recombination. The Eq.(11) for such a case yields

$$n\, a^2\, \psi'' + 2a\, \psi'\, (n - 0.5 + \delta/4) + (\delta - 1) = 0. \tag{18}$$

Then after the change of variable $\xi = \ln(a/a_*)$ the Eq. (18) yields

$$n\ddot{\psi} + (n - 1/2 + \epsilon/2)\, \dot{\psi} + \epsilon = 0, \dot{} = d/d\xi. \tag{19}$$

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[1]It is interesting to note that the mentioned exponents (13) are obtained in the recent paper [19] for the equation equivalent to Eq.(11). In $f(R)$ theory, the given exponents allow obtaining of the solutions for cold dark matter coinciding with ones in the classical Friedman model of the universe. Also, it has been shown in the paper [19] that these solutions are stable in the framework of HOGT. Let us notice we slightly change exponent n in the preset work to obtain weakly unstable solutions adequate to the observational data. Authors are grateful to authors of paper [19] kindly attracting our attention to their results.
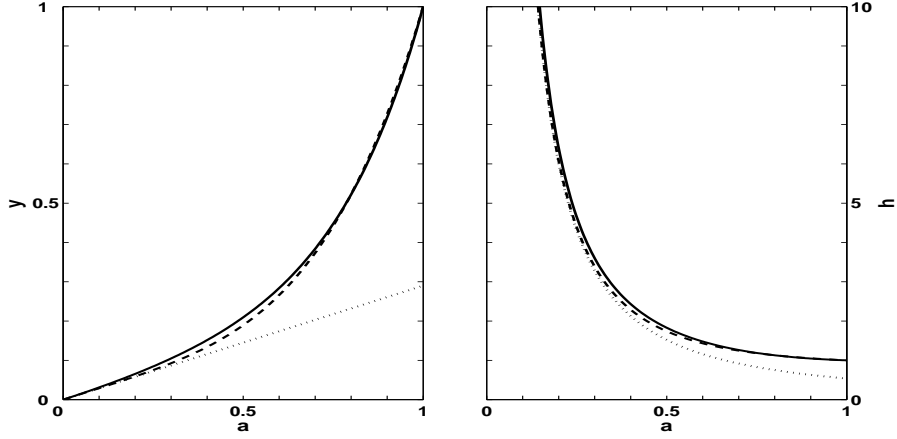
Figure 1: The evolution of the variable y (a left panel) and the Hubble parameter h (a right panel) with the scale factor a are represented by the solid line. The ΛCDM solution and the cold matter solution are presented by dashed and dotted line, respectively. Both graphs correspond to the case $\Omega_m = 0.29$.

The last equation has a solution

$$\psi = \left(C_1 + \frac{2\,\epsilon}{1-2n} \ln \left(\frac{a}{a_*} \right) \right) + C_2\, a^{(\frac{1}{2n}-1)}. \tag{20}$$

The analysis of (20) has shown that the requirement stated above is realized only for $\epsilon < 0$.

In this case, the modification of the exponent (17) is determined by a small positive correction

$$\delta n \simeq -2\,\epsilon\, n_1 (n_1 - 1)/(12 n_1 - 1). \tag{21}$$

The numerical analysis has shown that behavior of solutions is sensitive to small changes of ϵ at $a \to 1$. The last fact together with a choice of parameter β allows us to obtain a good enough correspondence with the observational data.

As an illustration we give the results for the set of parameters $n = 1.296$, $\beta = 0.467$ fixed according to $\Omega_m = 0.29, h_0 = 68$ (the best fit to the CMB+SNe data presented in the paper [5]). In a left panel of Fig. 1, a solid line represents the evolution of the variable y with a scale factor a. At the beginning, it coincides with the evolution of the dust model which is represented by a dotted line but further it deviates to the ΛCMD model represented by a dashed line. In a right panel, one can see evolution of the Hubble parameter with the scale factor a for the mentioned three models.

3.2   The behavior of the solution in future

The further expansion of the Universe at $a >> 1$ according to the Eq. (11) leads to the negligible effect of cold matter on the solution behavior. In this case, De Sitter solution
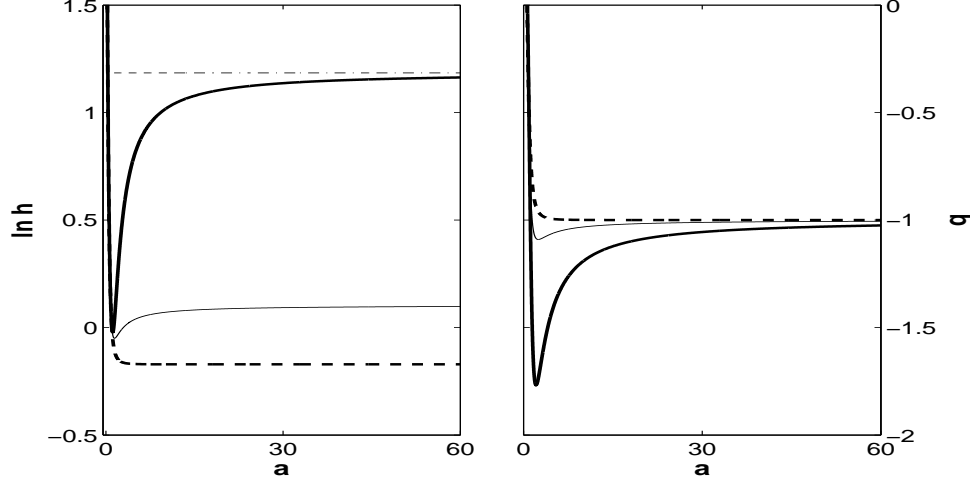
Figure 2: The evolution of the Hubble parameter h and the deceleration parameter q with a scale factor a. The thick solid lines represent graphs related to the $f(R)$-model while the dashed lines represent the evolution of the parameters of ΛCDM model. In the left panel the asymptotic solution for $f(R)$ model is shown by the dash-doted line.

$y = \Omega\, a^4$ is an asymptotic solution of Eq.(11). This solution corresponds to the constant Hubble parameter

$$h(a_{\longrightarrow\infty}) = \sqrt{\Omega} \tag{22}$$

with Ω defined from equality

$$3\beta(2n-1)(n-1)/2 = (4\Omega)^{(1-n)}/4 \tag{23}$$

The inflationary solution is stable in the process of evolution of the model. To show it, we shall look for the solution with perturbation in the form
$y = \Omega a^4(1+\Phi), \quad \Phi << 1$. This ansatz yields

$$n_1 a^2 \Phi'' + [(n_1-1)/2 + 3n_1^2]a\Phi' + 6(2n_1^2 - 3n_1 + 1)\Phi = 0 \tag{24}$$

The change of variable a to variable ξ (see Eq.(20)) yields

$$\ddot{\Phi} + A\dot{\Phi} + B\Phi = 0, \quad \dot{} = d/d\xi. \tag{25}$$

where coefficients are $A = [3n_1 - (1/n_1 + 1)/2] = 2.99$, $B = 6(2n_1 - 3 + 1/n_1) = 2.18$. This equation for perturbations have a damped solution indicating De Sitter solution to be stable. The numerical analysis has shown that De Sitter solution is an attractive solution.

The evolution of the Hubble parameter is represented by the thick solid line in the left panel of Fig. 2 for the case $\Omega_m = 0.29$. In contrast to the Hubble parameter of ΛCDM model (the dashed line) monotonically decreasing down to constant $\sqrt{\Omega_\Lambda}$, it reaches a

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minimum $h_m \approx 0.978$ at $a \approx 1.15$ and after that increases up to the asymptotic solution (22) which is represented in the figure by the dash-doted line. It is interesting to note that the formula for dimensionless parameter Hubble $h(z) = h(a)$ obtained from observational data in the paper [5] allows it's extrapolation in future $(a > 1)$. At the parameters mentioned in this paper, the formula for $h(a)$ also predicts minimum of the Hubble parameter at $a \approx 1.45$ which is equal to 0.951. This best fit of the paper [5] is shown in Fig.2 by the thin solid line. The deceleration parameter q represented in the right panel of Fig. 2 also passes a minimum and approaches to -1 with the growth of a. Therefore, we live in a transitional epoch between the classical Friedman cosmology and the De Sitter cosmology.

4   Results

Hereafter we will present the comparison of results of our model and the observational data.

As a set of observational data, the analysis of SNe+CMB data done in the paper [5] have been used. In that paper authors have reconstructed the resent history of the Universe on the base of SNe and CMB data in the model-independent way, only modelling DE by the hydrodynamical equation of state

$$p = w\rho, \quad w = (2q - 1)/(3 - \Omega_m/h^2). \tag{26}$$

The cited paper presents two conceptions of the analysis of observational data: the first of them is the best fit to the data which uses only the hydrodynamical describing of DE and does not impose restrictions on the values of Ω_m and h_0, while the second conception follows the priority of the concordance ΛCDM model, so authors of [5] put $\Omega_m = 0.27 \pm 0.04$ and $h_0 = 0.71 \pm 0.06$.

We will give the comparison of our results with both of them. Also we notice that the analogy of the "hydrodynamical" DE (26) is not so proper to the higher order gravity theories, hence one can expect the comparison over the values h and q to be more informative than over the value w.

It have been found in the paper [5] the best fit values are: $\Omega_m = 0.385, h_0 = 60$. In the model DE evolves in time strongly enough. In Fig. 3 one can see results (the evolution of the deceleration parameter and DE equation of state parameter with redshift) for analysis of paper [5]: the best fit is represented by the thin solid line, the 1σ and 2σ confidence levels are represented by the light and dark grey contours, respectively, and ΛCDM is represented by the dashed line. Also for given Ω_m and h_0 the results of the $f(R)$-model with $n = 1.2955\,(\delta n = 0.0002)$ and $\beta = 0.273$ for the "geometrical equation of state" parameter $w = (y - ay')/3(y - \Omega_m a)$ and deceleration parameter q are presented by the thick solid lines. In given $f(R)$-model the age of the Universe is 14.9 Hyr, the deceleration parameter is $q_0 = -0.91$ at present and the transition to acceleration occurs at $z = 0.38$. Similarly to results of [5], the $w_{DE} < 1$ at lower redshifts ($w_{DE0} = -1.53$), however , the evolution of equation of state of "geometrical DE" is more weak contrary to the results of [5].
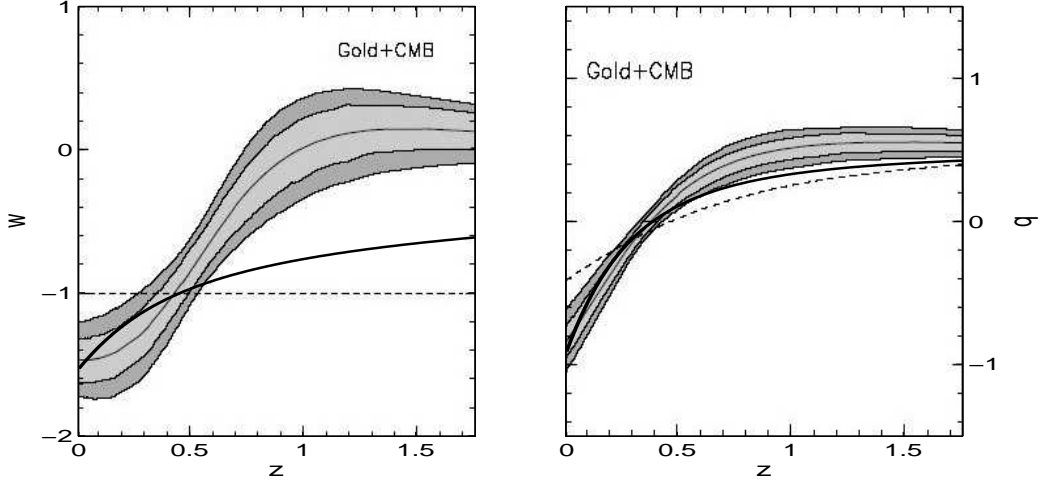
Figure 3: The comparison of results of the $f(R) \propto \beta R^n$ model to the analysis of the observational data quoted from the paper [5] by approbation of the authors. The evolution of the deceleration parameter with redshift is shown in a right panel and the variation of the parameter w of equation of state of DE is shown in a left panel. The best fit of SNe+CMB data with $\Omega_m = 0.385$, $h_0 = 0.60$ is represented by the thin solid line, the 1σ and 2σ confidence levels are represented by the light and dark grey contours, respectively, and ΛCDM is represented by the dashed line. For given Ω_m and h the $f(R)$-model is defined by parameters $n = 1.2955$ and $\beta = 0.273$ and represented here by the thick solid line.
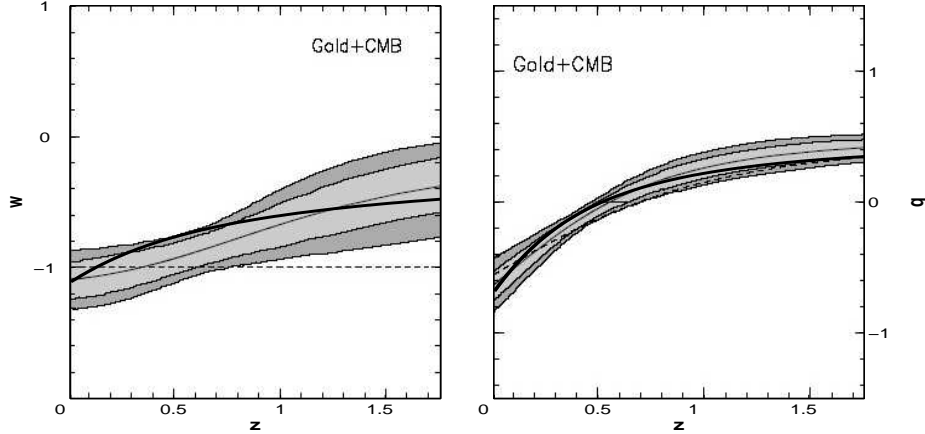

Figure 4: The comparison of results of $f(R) \propto R^n$ model with $n = 1.296$ and $\beta = 0.467$ (the thick solid line) to results of analysis of SNe+CMB data with $\Omega_m = 0.29$ done in [5]. The evolution of the deceleration parameter with redshift is shown in a right panel and variation of equation of state of DE is shown in a left panel. The best fit of SNe+CMB data in such case is represented by the thin solid line, the 1σ and 2σ confidence levels are represented by the light and dark grey contours, respectively, and ΛCDM is represented by the dashed line.

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However, if strong priors have been imposed on Ω_m and h_0 (i.e. the ΛCDM model priors: $\Omega_m = 0.27 \pm 0.04$ and $h_0 = 0.71 \pm 0.06$), the evolution of DE is extremely weak and in good agreement with the ΛCDM model. The best fit in the case is $\Omega_m = 0.29$. One can see good enough coincidence of our model and their analysis for parameters of the model $n = 1.296 \, (\delta n = 0.001)$ and $\beta = 0.467$ in figure 4. The deceleration parameter $q_0 = -0.683$ at present and the deceleration was changed by the acceleration at $z = 0.51$, $(q_0 = -0.63 \pm 0.12$ and $z = 0.57 \pm 0.07$ in the paper [5]). The age of the Universe in this case is 13.6 Hyr.

In Fig. 5 the comparison of the evolution of the Hubble parameter for $f(R)$-model to ΛCDM model and the best fit of the observational data analysis is presented. One can see the results for the cases with $\Omega_m = 0.385$ and $\Omega_m = 0.29$ in a left and a right panel, respectively.

Thus, the $f(R) \propto \beta R^n$-model with parameters β and n chosen according to the principles mentioned in Introduction describes the evolution of the Universe quite corresponding to the SNe+CMB data.
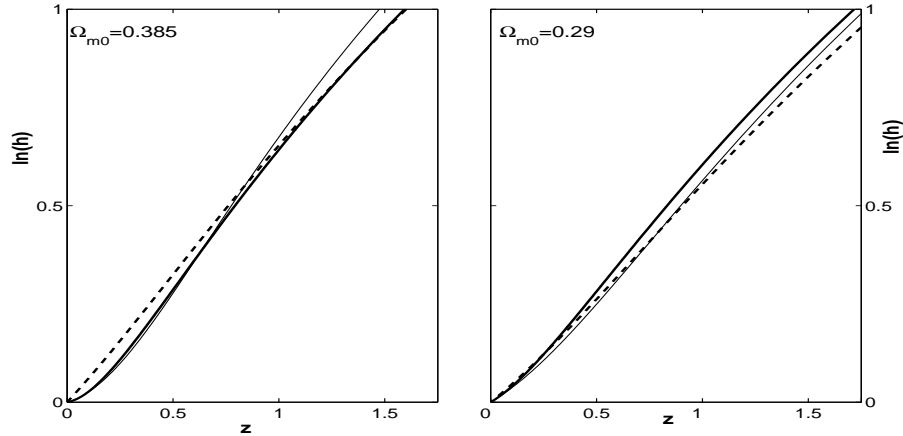

Figure 5: The evolution of the Hubble parameter with redshift. The $f(R)$-model and the ΛCDM model are shown by the thick solid and the dashed lines, while the best fit of the observational data according to [5] is represented by the thin solid line. The case for $\Omega_m = 0.385$, $h_0 = 0.6$ is shown in the left panel, and the case $\Omega_m = 0.29$, $h_0 = 0.68$ is shown in the right panel.

5   Acknowledgements

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